Mathematical connections in combinatorial enumeration

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Outline

1. Enumerative Combinatorics
   What We Count and How We Count It
   The Generating Function

2. Applying Complex Analysis
   Cauchy’s Formula and Coefficient Extraction
   Pemantle-Wilson Analysis, Part I

3. Enter Algebra, Geometry and Topology
   The Singular Variety
   Pemantle-Wilson Analysis, Part II
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What is Enumerative Combinatorics?

Enumerative Combinatorics is the mathematics of counting — typically counting discrete mathematical objects.

A class of such objects is usually partitioned by a set properties, and the task is to count the sizes of these partitions.

These properties are typically each indexed by natural numbers.
Examples
The Fibonacci Numbers

You start with a pair of rabbits, born at the start of month 1. A pair of rabbits mates at the beginning of every month, starting when they are one month old. One month after mating, each pregnant rabbit gives birth to a new rabbit pair. Rabbits never die. How many pairs of rabbits $f_n$ are there at the start of month $n$?

**Answer:** The Fibonacci numbers:

$$f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, \ldots$$

$$f_n = f_{n-1} + f_{n-2} \text{ for } n > 2$$
Examples, cotd.
The Binomial Coefficients

On an $r \times s$ grid, how many ways $b_{r,s}$ are there to travel from the lower left corner of the grid to the upper right corner, stepping only up and right.

**Answer:** The binomial coefficients.

$$b_{r,s} = \binom{r+s}{r}$$

**Figure:** The $\binom{4}{2} = 6$ ways of navigating a $2 \times 2$ board.
Examples, cotd.

The Delannoy Numbers

On an $r \times s$ grid, how many ways $d_{r,s}$ are there to travel from the lower left corner of the grid to the upper right corner, stepping only up, right or diagonally up and right.

**Answer:** The Delannoy numbers.

**Figure:** An example path on a $3 \times 5$ grid. Note: $d_{3,5} = 231$. 
What Do We Mean By Counting?

There are many different things we may mean by counting, including

- Obtaining a closed form counting function.
- Obtaining a recursive counting function.
- Obtaining an asymptotic counting function.

It is the third of these possibilities with which we will be concerned.
Asymptotic Counting

**Definition**

Two functions $f$ and $g$ are said to be *asymptotic*, and we write $f \sim g$, if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.$$ 

Thus, to count asymptotically means to come up with an approximating formula whose relative error goes to zero.
An Example of Asymptotics

**Example**

For the (diagonal) binomial coefficients, we have

\[
\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}
\]

A question you might ask yourself: what earthly right does \( \pi \) have showing up in that formula!? 
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The Generating Function

An Overview

Generating functions are the main tool we will use to count asymptotically. So what are they?

“A generating function is a clothesline on which we hang up a sequence of numbers for display”.

– Herbert Wilf

To a sequence of numbers, we associate an algebraic object — a formal power series — in which the sequence is embedded.
The Definition

Definition

The (ordinary) generating function associated to a (bivariate) sequence $a_{r,s}$ of numbers is the formal power series

$$F(x, y) = \sum_{r,s=0}^{\infty} a_{r,s} x^r y^s.$$  

Idea: Sometimes this defines a holomorphic function (near the origin). Thus we may be able to use complex analysis to study the coefficients.
Example: The Binomial Coefficients

Denoting the generating function by

\[ F(x, y) = \sum_{r,s} b_{r,s} x^r y^s, \]

the recurrence relation \( b_{r,s} = b_{r-1,s} + b_{r,s-1} \) (for \( r, s \geq 1 \)) leads to the relation

\[ F(x, y)(1 - x - y) = 1, \]

and so we have

\[ F(x, y) = \frac{1}{1 - x - y}. \]
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Cauchy’s Formula

**Theorem**

Let $U \subseteq \mathbb{C}$ be an open set on which a function $F$ is holomorphic, and let $D \subseteq U$ be a closed disc. Then for $x$ in the interior of $D$,

$$F(x) = \frac{1}{2\pi i} \int_{\partial D} \frac{F(z)}{z - x} \, dz.$$ 

Note: this is actually just a form of Stokes’ Theorem (the fundamental theorem of calculus):

$$\int_D d\omega = \int_{\partial D} \omega,$$

where the “differential” is actually a Dirac delta function.
Differentiating Cauchy’s Formula

From Cauchy’s formula we have

$$\frac{d^n}{dx^n} F(x) = \frac{d^n}{dx^n} \frac{1}{2\pi i} \int_{\partial D} \frac{F(z)}{z-x} \, dz$$

Differentiating under the integral sign, this yields

$$F^{(n)}(x) = \frac{n!}{2\pi i} \int_{\partial D} \frac{F(z)}{(z-x)^{n+1}} \, dz$$

$$\frac{F^{(n)}(x)}{n!} = \frac{1}{2\pi i} \int_{\partial D} \frac{F(z)}{(z-x)^{n+1}} \, dz$$

But the left hand side of this equation is just the formula for the $n^{th}$ coefficient in the Taylor expansion of $F$ at $x$. 
Coefficient Formula (one variable)

**Theorem**

Let \( F(z) = \sum_{n=0}^{\infty} a_n z^n \) be a function holomorphic in a neighborhood of the origin. Then we have

\[
a_n = \frac{1}{2\pi i} \int_{\partial D} \frac{F(z)}{z^{n+1}} \, dz,
\]

where \( \partial D \) is a sufficiently small circle about the origin.

Computation of the coefficients reduces to computation of an integral.
Coefficient Formula (two variables)

**Theorem**

Let \( F(x, y) = \sum_{r,s=0}^{\infty} a_{r,s} x^r y^s \) be a function holomorphic in a neighborhood of the origin. Then we have

\[
a_{r,s} = \frac{1}{(2\pi i)^2} \int_\mathcal{T} \int \frac{F(x, y)}{x^{r+1} y^{s+1}} \, dx \, dy
\]

where \( \mathcal{T} \) is the product of sufficiently small circles about the origin in the \( x \) and \( y \) planes.

This follows by iterated integration.

Note: similar formulas exist for any number of variables.
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The Situation

Pemantle-Wilson deals only with the case where the generating function is rational. Specifically, we have:

- A sequence $a_{r,s}$ that we wish to count
- The sequence’s generating function
  $$F(x, y) = \sum_{r,s=0}^{\infty} a_{r,s} x^r y^s.$$  
- We assume that $F$ is holomorphic near the origin, where it takes the form $P/Q$ for some polynomials $P$ and $Q$.

For simplification, we will only examine the case when $r = s = n$, with $n \to \infty$. 
Motivating Idea

We must analyze the integral

$$\int \int_T \frac{P}{x^{n+1}y^{n+1}Q} \; dx \; dy$$

as $n \to \infty$.

We see that the asymptotic magnitude of the integrand is governed by $|x|$ and $|y|$.

Idea: Push the domain of integration $T$ out toward infinity, where the integrand’s magnitude is smaller.
An Outline in One Variable

- Push contour toward infinity (doesn’t change integral, thanks to Stokes’ Theorem).
- Push contour around singularities.
- Integral far from origin is asymptotically negligible.
- Integral near singularities reduces to the value of a residue function at the point.
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What Happens in Two Variables?

In more than one variable, the outline remains essentially unchanged except for the last step.

In the final step we must compute the integral of a residue function over a cycle on the singular set (reduces problem by one dimension).

Thus we must understand the geometric structure of the singular set, specifically where $Q = 0$. 
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Algebraic Geometry

Algebraic Geometry is the study of geometric problems as approached using an algebraic framework.

Of prime interest to classical algebraic geometry: The correspondence between a set $S$ of polynomials and the vanishing set $\mathcal{V}$ of $S$, called an algebraic variety.

We are interested in the variety $\mathcal{V}_Q$ of the vanishing set (in $\mathbb{C}^2$) of a single polynomial $Q(x, y)$. 
An Example: Bicolored Supertrees

As an example, take

\[ Q(x, y) = x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2 \]

What does \( V_Q \) look like?

\( Q \) is quadratic in \( y \), so we can solve

\[ y = \frac{-x^2 + x^3 - 2 + \sqrt{x^4 + 4x^2 - 4x^3 + 4}}{x^5} \]

For most values of \( x \) we get two values of \( y \). As parameterized by \( x \), \( V_Q \) basically looks like two sheets that connect together in some complicated way.
The Singular Variety

An Example, cotd.

Visualization

Can show that $V_Q$ is a smooth surface, but it is a two (real) dimensional surface in a four (real) dimensional space. How do we picture that?

Idea: throw out one of the dimensions! (ignore $\Im y$).
An Example, cotd.

Visualization

Can show that $\mathcal{V}_Q$ is a smooth surface, but it is a two (real) dimensional surface in a four (real) dimensional space. How do we picture that?

Idea: throw out one of the dimensions! (ignore $\Im y$).

**Figure:** A depiction of $\mathcal{V}_Q$ in three dimensions.
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The Last Step

What remains: to compute the (one-dimensional) integral of a residue function along a cycle on $\mathcal{V}_Q$.

The asymptotic magnitude of the integrand is still governed by the magnitudes of $x$ and $y$.

Idea (again): Manipulate the contour of integration to minimize the size of the integrand. We define a height function:

$$h(x, y) = -\ln |x| - \ln |y|$$

and attempt to minimize the maximum height along the contour of integration.
The Last Step, cotd.

The idea: flow contour down to lower heights (minimize the maximum height on the contour)

Where might this get stuck? Saddle points of the height function!

This is very good! Near saddle points, we can apply the method of steepest descent (asymptotic formulae).
An Example: Bicolored Supertrees

The variety $\mathcal{V}_Q$; now colored by the height function. Red indicates high points and violet indicates low points.

**Figure:** Manipulating the contour of integration, bicolored supertree example
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*Figure*: Manipulating the contour of integration, bicolored supertree example
Homology and Morse Theory

We are hitting upon two very important topics in topology.

- **Homology Theory**: We replace the contour of integration by one which is in some sense equivalent. Homology captures this notion of equivalence.

- **Morse Theory**: We want to find an equivalent contour whose height is maximized at a saddle point. Morse Theory deals directly with calculating homology, and how homology is affected by critical points of a height function.
Finishing our Computation

With the height along the contour of integration maximized at saddle points, the integral locally breaks into:

- saddle-point integrals near these points, and
- negligible pieces away from these points.

Finally, we apply saddle point integration formulas to obtain asymptotics.
Saddle Point Integration

A simple example theorem:

**Theorem**

*For any \( \varepsilon > 0 \), we have:*

\[
\int_{-\varepsilon}^{\varepsilon} e^{-\frac{nx^2}{2}} \, dx \sim \sqrt{\frac{2\pi}{n}}
\]

Note: theorems exist for integrands of the more general form

\[A(x)e^{nh(x)}\]

where \( h'(0) = 0 \) and \( h(x) \) is maximized at \( x = 0 \).
Closing Points

Future Research:

• How might we automate this task (two variable case)?
• How can we perform this analysis in three and more variables (the topology gets much more complicated at the critical values of the height function)?

Connections in Mathematics:

“Solving a problem simply means representing it so as to make the solution transparent.”

– Herbert Simon
Further Reading

- Pemantle, R. and Wilson, M.  
  Twenty combinatorial examples of asymptotics derived from multivariate generating functions.  
  *SIAM Review*, vol. 50, pages 199-272.

- DeVries, T.  
  A case study in bivariate singularity analysis.  