Counting by Climbing
Based on joint work with R. Pemantle and J. van der Hoeven

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Outline

1. Climbing Alien Terrain
   - Background
   - The Algorithm

2. Translating into Algebraic Geometry
   - The Setup
   - Implementation

3. Topology and Homology
   - Topology of the Variety
   - Combinatorics
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Main Interest: Enumerative Combinatorics

Enumerative Combinatorics is the mathematics of counting — typically counting discrete mathematical objects.

A class of such objects is usually partitioned by a set of properties, and the task is to count the sizes of these partitions.

These properties are typically each indexed by natural numbers.
Example
The Delannoy Numbers
On an $r \times s$ grid, how many ways $d_{r,s}$ are there to travel from the lower left corner of the grid to the upper right corner, stepping only up, right or diagonally up and right.

Answer: The Delannoy numbers.

Figure: An example path on a $3 \times 5$ grid. Note: $d_{3,5} = 231$. 
John Q. Combinatorialist’s Claim

How do we compute a quantity like $d_{r,s}$? According to John, we can solve this problem by sending exploration robots to climb mountains on an exotic alien world in a 5-dim universe.

This world – which we shall call $\mathcal{V}$ – and its universe are exotic for a number of reasons.

- The planet is two-dimensional (all surface).
- Gravity acts uniformly to push objects downward in the fifth dimension. We think of the fifth-dimensional coordinate as indicating “height” in the universe.
- The terrain of $\mathcal{V}$ has no local maxima or minima, and contains finitely many saddle points (denoted by the set $\Sigma$).
- As the height increases, one of two forces ($x$ force or $y$ force) begins to dominate.
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The Explorer Bot Algorithm

For each saddle point $\sigma \in \Sigma$ (in decreasing order of height):

- Examine the surface of $\mathcal{V}$ near $\sigma$ to determine the number of different ascent directions.
- For each ascent direction at $\sigma$, do the following:
  - Take an ascent step in that direction.
  - Repeatedly: Examine the local geometry to determine the direction of steepest ascent, and then make an ascent step in that direction. Terminate when, either:
    - The height is sufficiently high to determine which force ($x$ or $y$) dominates, or
    - The path climbs to (near) a previously explored saddle point.

Report the findings: which ascent paths lead to peaks exhibiting which force? These findings enable us to count.
The Difficulties

Difficulty #1: Tread lightly...

There are a number of difficulties in implementing the preceding algorithm. The first: making sure that the explorer bot takes steps correctly.

There are (at least) two things to worry about:

1. There are four degrees of freedom for movement, but only two degrees of freedom if you wish to stay on the planet’s surface. Don’t step off into oblivion!

2. The planet may be folded in space in very strange ways. A step should be made along the surface of $\mathcal{V}$, never bridging a gap through space.
The Difficulties

Difficulty #2: It’s about precision...

The second difficulty: finite precision. By dealing with a computer algorithm, we are forced (to some degree) to approximate computations up to a finite precision.

This causes a number of problems, including:

1. How do we represent the explorer bot’s exact location on the planet?
2. We may need to approximate the steepest ascent direction. How can we guarantee that the approximate direction chosen leads to a genuine ascent? (A problem of discretization.)
The Difficulties

Difficulty #3: When should the algorithm terminate?

Finally, what is our exact criterion for when an ascent path should terminate?

Some specific questions:

1. How do we know that the explorer bot has climbed high enough to determine whether the $x$ force or the $y$ force will dominate? Maybe it will swap as we climb higher.

2. How do we know the explorer bot will even continue to climb? Maybe it will climb by smaller and smaller amounts, approaching some height asymptote.

The answer to these questions lies in the mathematical translation of our problem, and in our choice of data type.
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The Real Exploration

Given a polynomial $Q \in \mathbb{Z}[x, y]$ with $Q(0, 0) \neq 0$, we define the variety

$$\mathcal{V}_Q = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\},$$

which we assume to be smooth. We then define

$$\mathcal{V} = \mathcal{V}_Q \setminus \{(x, y) \in \mathbb{C}^2 : xy = 0\}.$$

Height along $\mathcal{V}$ is governed by the height function

$$h(x, y) = -\hat{r} \log |x| - \hat{s} \log |y| = \Re H(x, y)$$

where $H(x, y) = -\hat{r} \log x - \hat{s} \log y$ (a locally-defined, complex-analytic function on $\mathcal{V}$) and $\hat{r}$ and $\hat{s}$ are positive constants.
Forces and Saddle Points

We imagine that the $x$ force at a point $(x, y) \in V$ to be inversely proportional to $|x|$. Similarly for the $y$ force. Thus the information we are actually tracking is whether each ascent path approaches $x = 0$ or $y = 0$.

The set $\Sigma$ of saddle points can be found by computing where (along $\mathcal{V}$) $\nabla H \parallel \nabla Q$, i.e. by simultaneously solving

$$Q(x, y) = 0,$$

and

$$\hat{r} y \frac{\partial Q}{\partial y} (x, y) - \hat{s} x \frac{\partial Q}{\partial x} (x, y) = 0.$$
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Solving Many Problems

Ball Arithmetic

We perform all our arithmetic/computations on so-called ball numbers

\[ B_r(c) = \{ z \in \mathbb{C} : |z - c| < r \}, \quad c \in \mathbb{Q}[i], \quad r \in \mathbb{Q} \]

Given some \( B_r(c) \) as an input to some computable function \( f(z) \), the output is a ball \( B_\rho(\gamma) \) such that for any \( z_0 \in B_r(c) \), we have \( f(z_0) \in B_\rho(\gamma) \).

We have two main points of view:

1. \( B_r(c) \) is an approximation of our actual input.
2. Ball arithmetic performs many computations at once.

Ball arithmetic is implemented in Mathemagix (by Joris van der Hoeven), and in a very powerful way.
Applying Ball Arithmetic

Application #1: Staying on the Surface

Start at some \((x_0, y_0) \in \mathcal{V}\), for \(x_0\) and \(y_0\) represented by ball numbers of extremely small radius.

Assume (check) that \(\mathcal{V}\) is locally parameterizable by \(x\) near \((x_0, y_0)\) (i.e. \(\frac{\partial Q}{\partial y} \neq 0\)).

Let \(B\) denote the expansion of \(x_0\) to a larger ball of some radius \(r\). Try to solve \(Q(B, y) = 0\) as a ball number for \(y\). This will return a list of ball numbers (except for those that overlap or go to infinity). Decrease \(r\) until you produce a solution \(B'\) that overlaps with \(y_0\).

Step to a new \(x_1 \subset B\). The corresponding \(y_1\) is the solution in \(y\) to \(Q(x_1, y) = 0\) that lies in \(B'\).
Applying Ball Arithmetic

Application #2: Guaranteeing Ascent

Local to a point \((x_0, y_0) \in \mathcal{V}\), assume (check) that \(\mathcal{V}\) can be parameterized by \(x\). We can write

\[
\left. h \right|_{\mathcal{V}} = \Re \left( c_0 + c_1 (x - x_0) + O((x - x_0)^2) \right) \quad \text{as} \quad x \to x_0.
\]

Locally, have an ascent whenever

\[
\arg(x - x_0) \in (-\arg(c_1) - \pi/2, -\arg(c_1) + \pi/2),
\]

where

\[
c_1 = \frac{\partial H}{\partial x} - \frac{\partial H}{\partial y} \left( \frac{\partial Q}{\partial x} / \frac{\partial Q}{\partial y} \right).
\]

Expand \(x_0\) to a ball \(B\) and compute a corresponding ball \(B'\) for \(y_0\). Compute \(c_1\) at \((B, B')\). If it is contained in a half-plane, we can find a single direction corresponding to an ascent within the entirety of \(B\).

If not in half-plane, shrink \(B\) and try again.
Termination

How do we know when a particular ascent must approach $x = 0$? When it must approach $y = 0$?
Given that $Q(0, 0) \neq 0$, there must be an $\varepsilon > 0$ such that $Q(x, y) \neq 0$ when $|x|, |y| \leq \varepsilon$. Pick $M$ sufficiently large so that

$$h(x, y) > M \Rightarrow |x| < \varepsilon \text{ or } |y| < \varepsilon.$$  

Then as soon as an ascent path rises to a height greater than $M$, exactly one of $|x|$ or $|y|$ will be small (less than $\varepsilon$). The order of $|x|$ and $|y|$ will never change from that point onward.

How do we know that the path will reach height $M$? Apply a compactness argument away from the saddle points, the coordinate axes and infinity.
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Analysis of the Topology

So what have we really done? Claim: We have enough information to recover the topology of $\mathcal{V}$, and to discuss the homology of of a particularly important cycle.

The key is to examine the space

$$\mathcal{V}^M = \{(x, y) \in \mathcal{V} : h(x, y) > M\}$$

in two steps. First, for $M$ sufficiently large. Second, we use Morse Theory to see how the topology changes as $M$ decreases, as more of $\mathcal{V}$ is unveiled.
Step 1: Describing \( \mathcal{V} \) at Large Height

\( h(x, y) \) is large when \( x \) or \( y \) is small, so we examine \( h \) near \( x = 0 \) and \( y = 0 \).

**Theorem (Puiseux expansion)**

Near \( x = 0 \), each branch of \( Q(x, y) = 0 \) admits a fractional series expansion:

\[
y(x) = \sum_{j \geq j_0} c_j x^{i/k}
\]

for some \( k \in \mathbb{N} \) and some \( j_0 \in \mathbb{Z} \).

Note: near \( x = 0 \), each branch looks like the solution set of \( y = x^{1/k} \).
Step 1, cotd.

So each branch has the form \( y \sim c_{j_0} x^{i_0/k} \) as \( x \to 0 \). So near \( x = 0 \):

\[
h(x, y(x)) \sim -\hat{r} \log |x| - \hat{s} \log \left| c_{j_0} x^{i_0/k} \right|
\]

\[
= (-\hat{r} - \frac{j_0}{k} \hat{s}) \log |x| - \hat{s} \log |c_{j_0}|
\]

And we see that \( h \) goes to \( \infty \), \( -\infty \) or a constant according to whether \((-\hat{r} - \frac{j_0}{k} \hat{s})\) is less than, greater than, or equal to 0.

Note: we toss out examples where the height is bounded near \( x = 0 \) or \( y = 0 \).
Step 1: a Picture

Restricting our attention to points on $V_Q$ of very large height, we end up with a set of punctured disks (one for each branch) due to $|x|$ being small, and similarly due to small $|y|$.

Picture: $V^{>M}$ for $M$ sufficiently large. We refer to the small $|x|$ disks as $x$-objects and to the small $|y|$ disks as $y$-objects.
Morse Theory, Part I

What happens to $\mathcal{V}^M$ as we decrease the value of $M$?

**Theorem (Morse Theorem I)**

Assuming $h$ has no critical points with values in the interval $(M', M)$, then $\mathcal{V}^{M'}$ is topologically equivalent to $\mathcal{V}^M$.

In fact, $\mathcal{V}^{M'}$ deformation retracts onto $\mathcal{V}^M$.

(Note: requires that $h$ is proper.) The idea is that, in the absence of critical/saddle points, $\mathcal{V}^M$ flows down to $\mathcal{V}^{M'}$. 
Morse Theory, Part II

What happens as $M$ passes a critical value for $h$?

- Locally, near a critical point, $h|_\mathcal{V}$ looks like
  \[ h(z) = \Re(c + z^n) \]
  for some $n \geq 2$.
- We lower the value of $M$, and $n$ components of $\mathcal{V}$ draw nearer.
- When $M$ reaches the height of the critical point, $n$ components join.
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Piecing together the Variety

Thus $\mathcal{V}$ is constructed by gluing together punctured disks. And thanks to the “explorer bots,” we know how this works!

Specifically, near any saddle point $\sigma \in \Sigma$, the ascent paths enter into the $n$ components that become fused as $M$ is decreased below $h(\sigma)$.

Now why do we care about the topology of some variety $\mathcal{V}_Q$? And why did we track the dominant “force” applying to each ascent path? It is finally time to turn back to Combinatorics.
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The Generating Function

Definition

The (ordinary) *generating function* associated to a (bivariate) sequence \(d_{r,s}\) of numbers is the formal power series

\[
F(x, y) = \sum_{r,s=0}^{\infty} d_{r,s} x^r y^s.
\]

Idea: Sometimes this defines a holomorphic function (near the origin). Thus we may be able to use complex analysis to study the coefficients.
Example: The Delannoy Numbers

The recurrence relation \( d_{r,s} = d_{r-1,s} + d_{r,s-1} + d_{r-1,s-1} \) (for \( r, s \geq 1 \)) leads to the relation

\[
F(x, y)(1 - x - y - xy) = 1,
\]

and so we have

\[
F(x, y) = \frac{1}{1 - x - y - xy}.
\]
Coefficient Formula (two variables)

**Theorem**

Let \( F(x, y) = \sum_{r,s=0}^{\infty} d_{r,s} x^r y^s \) be a function holomorphic in a neighborhood of the origin. Then we have

\[
d_{r,s} = \frac{1}{(2\pi i)^2} \int_T \frac{F(x, y)}{x^{r+1} y^{s+1}} \, dx \wedge dy
\]

where \( T \) is the product of sufficiently small circles about the origin in the \( x \) and \( y \) planes.

This is simply an iterated form of the residue theorem.

Note: similar formulas exist for any number of variables.
The Setup

We assume that $F$ converges to a rational function near the origin, i.e. $F = P/Q$ for some polynomials $P$ and $Q$. This is the $Q$ of our previous analysis.

We will examine $d_{r,s}$ as $r$ and $s$ go to infinity. Specifically, we fix constants $\hat{r}$ and $\hat{s}$ and set

$$r = n\hat{r}, \quad s = n\hat{s}, \quad n \to \infty$$

Essentially, we are picking a direction to take the asymptotics (fixing a ratio of $r$ to $s$).

Note: a more detailed analysis allows simply for $r \sim n\hat{r}$ and $s \sim n\hat{s}$ (in most cases).
Motivating Idea

We must analyze the integral

$$\int_T \frac{P}{x^{n\hat{r}+1}y^{n\hat{s}+1}} Q \ dx \wedge dy = \int_T \frac{P}{xyQ} e^{nH(x,y)} \ dx \wedge dy$$

as $n \to \infty$.

We see that the asymptotic magnitude of the integrand is governed by $h(x, y)$.

Idea: Push the domain of integration $T$ out toward infinity, where the integrand’s magnitude is smaller.
An Outline in One Variable

- Push contour toward infinity.
- Push contour around singularities.
- Integral far from origin is asymptotically negligible.
- Integral near singularities reduces to the value of a residue function at the point.
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What Happens in Two Variables?

In more than one variable, we end up with the integral of a residue form over a 1-cycle on the singular set (rather than a sum of residue values at isolated points). Why?

- Imagine pushing $T$ through $\mathcal{V}_Q$ by a homotopy $\mathcal{H}$. Assuming transverse intersection, $\mathcal{H}$ intersects $\mathcal{V}_Q$ in a 1-cycle $C$, called the intersection cycle.

- Instead, push $T$ around $C$, picking up an integral on a tubular neighborhood of $C$.

- Locally this is an iterated integral. First integrate along a circle in the normal slice to $\mathcal{V}_Q$ (ordinary residue theorem), what remains is an integral of a residue 1-form along $C$ itself.
A Formula for the Residue Form

Away from the points where \( x, y \) and \( \frac{\partial Q}{\partial y} \) vanish, we can represent the residue form by

\[
\text{Res} \left( \frac{P}{x^{r+1}y^{s+1}Q} \, dx \wedge dy \right) = \frac{-P}{x^{r+1}y^{s+1} \frac{\partial Q}{\partial y}} \, dx
\]

And we must compute

\[
d_{r,s} = \frac{1}{2\pi i} \int_C \frac{-P}{x^{r+1}y^{s+1} \frac{\partial Q}{\partial y}} \, dx = \frac{1}{2\pi i} \int_C xy \frac{\partial Q}{\partial y} e^{nH} \, dx
\]

Idea: Manipulate \( C \) so as to minimize the magnitude of the integrand. This is what requires an understanding of the topology of \( \mathcal{V} \).
The Intersection Cycle

The homotopy $\mathcal{H}$ that we use to get the intersection cycle $C$ is to fix $|x| = \varepsilon$ small, and let $|y|$ grow large. This results in small cycles in the $x$ disks.

![Diagram of intersecting cycles](image)

The algorithm gives us information about the homology class of $C$, and how to choose a homologous cycle appropriate for integration.
Integrate: Saddle Point Theorem

Replace $C$ by a homologous cycle along which height is maximized at a saddle, then use the following theorem near the saddle:

**Theorem**

If $\phi'(0) = 0$ and $\gamma : [-\varepsilon, \varepsilon] \to \mathbb{C}$ is any smooth curve with $\gamma(0) = 0$, $\gamma'(0) \neq 0$ and such that $\Re[\phi(\gamma(t)) - \phi(0)] \leq 0$ (equality only at $t = 0$), then

$$\int_{\gamma|_{[0,\varepsilon]}} A(x)e^{n\phi(x)} \, dx \sim e^{n\phi(0)} \sum_{j=l}^{\infty} \frac{a_j}{k} \Gamma \left( \frac{1+j}{k} \right) n^{-\frac{(1+j)}{k}},$$

for some $a_j$ (computable from the coefficients of $A$ and $\phi$).

Think of this as a juiced up Gaussian integral.
Outlook

Some future work/questions include:

- How do we deal with cases where the height is bounded as $x$ or $y$ goes to 0 (so as $y$ or $x$ goes to $\infty$)? What about saddle points at $y$ or $x = \infty$?

- How can we perform this analysis in three and more variables, when the topology of critical points can be more varied (no longer a simple saddle point)? Solving three variable rational asymptotics would solve two variable algebraic asymptotics (by Safonov’s algorithm).
Further Reading

DeVries, T., van der Hoeven, J. and Pemantle, R.
Automatic asymptotics for coefficients of smooth, bivariate rational functions.

DeVries, T.
Algorithms for bivariate singularity analysis.
http://www.haverford.edu/math/tdevries/