Print your name: ____________________________

Sign your name: ____________________________

*I accept full responsibility under the Haverford Honor System for my conduct on this exam.*

Math 392 Final Exam Spring 2013

You are allowed up to 5 (consecutive) days (from start to finish) to work on the exam, subject to the following condition: Your exam must be completed and handed in by 12:01pm EDT on Saturday, May 11th, 2013. **If your exam is not in my possession or in my office with the honor pledge signed by this time, you will receive a 0 on the exam.** This is an open book exam. Specifically, you may consult the following resources before and during the exam: both course textbooks, your own notes/homeworks, and the solutions posted to Moodle. You may not consult any other resources (including Wikipedia, computer algebra systems, etc.). You may not consult with anyone else on this exam (unless you are contacting the professor for clarification). Be sure to justify all of your solutions; a correct solution without full justification will not earn full credit, while partial credit may be awarded for sufficient progress toward a correct solution. The questions count different amounts, as indicated. If there are any questions about these instructions or the exam, please ask.

**Time: Up to 5 days**

**Total Points: 61 points**

**Points**

1. ____________ / 7 points.

2. ____________ / 8 points.

3. ____________ / 9 points.

4. ____________ / 6 points.

5. ____________ / 9 points.

6. ____________ / 7 points.

7. ____________ / 8 points.

8. ____________ / 7 points.

Total ____________ / 61 points.
1. [7 points] A Tale of Two Polynomials.

Let \( f : A \rightarrow \mathbb{C} \) be analytic on some region \( A \), and assume that
\[
\text{Re } f(x + iy) = u(x, y)
\]
is a polynomial in \( x \) and \( y \). Show that \( f(z) \) is a polynomial in \( z \).

2. [8 points] Improper Integrals.

Compute the integral
\[
\int_{0}^{\infty} \frac{1}{\sqrt{x}(1 + x^2)} \, dx.
\]

3. [9 points] Residues and Integration.

Define the function
\[
f(z) = \frac{e^{i\pi z} - 1 - z}{z^4 - z^2}.
\]
(a) Compute the residues of \( f(z) \) at all of its singular points.
(b) Are there any curves \( \gamma : [0, 1] \rightarrow \mathbb{C} \) (not passing through any of the singular points of \( f \)) for which both
(1) \( I(\gamma; z_0) \neq 0 \) for some singular point \( z_0 \) of \( f \), and
(2) \( \int_{\gamma} f(z) \, dz = 0 \)?
If not, justify why not. If so, describe such a curve.


Suppose that \( f : D(0; 1) \rightarrow \mathbb{C} \) is analytic, and that there is a constant \( M > 0 \) such that
\[
|f^{(k)}(0)| \leq M^k \quad \text{for all } k.
\]
Show that there is an entire function \( g : \mathbb{C} \rightarrow \mathbb{C} \) that agrees with \( f \) on the unit disk.

5. [9 points] Combinatorics.

Denote by \( \mathcal{W} \) the class of words on the alphabet \( \mathcal{A} = \{a, b, c\} \) subject to the condition that the letter \( c \) only occurs in pairs. That is, the length of any run of consecutive \( cs \) appearing in any word must be even. So for example, we have
\[
\text{abaccbbacccccab} \in \mathcal{W},
\]
while
\[
\text{cabccc} \notin \mathcal{W}.
\]
As usual, the size of any given word is equal to its length.
(a) Compute the ordinary generating function \( W(z) \) for the combinatorial class \( \mathcal{W} \).
(b) Using your answer to part (a), compute an asymptotic formula for the total number of words \( w_n \) of size \( n \). Your formula should be of the form
\[
w_n \sim cA^n
\]
for some \( c, A \in \mathbb{R} \).

6. [7 points] Uniform Continuity.

Let \( f \) be an analytic function with domain \( D(0; 2) \) on which \(|f(z)| \leq 10\). Find a value of \( \delta > 0 \) independent of \( f \) such that for any \( z_1, z_2 \in D(0; 1) \) we have
\[
|z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \frac{1}{50}.
\]

*Hint: Use the Cauchy Integral Formula.*
7. [8 points] The Dirichlet Problem.
Find a formula for determining the temperature $T(x, y)$ in the region $\mathcal{R}$ depicted below, assuming that the temperature along the boundary of the region is held constant at the values indicated: $T(x, y) = 212$ degrees along the top edge of $\mathcal{R}$, $T(x, y) = 32$ degrees along the bottom edge of $\mathcal{R}$.

Let $f : \mathbb{C} \to \mathbb{C}$ be entire and bijective. Show that $f(z) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$. 